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Left Ideal Axioms for Non-associative Rings*

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1. INTRODUCTION

In non-associative ring theory the lattice of left ideals has not proven nearly as useful a tool as in the associative theory. A partial reason for this is that given an element x in a non-associative ring R, Rx need not be a left ideal of R.

In this paper we wish to investigate, under certain restrictions, some axioms for non-associative rings R that imply Rx is a left ideal for every x in the ring R. If R has an identity element, the latter property is equivalent to the statement:

$$\text{For every } x \in R, \qquad R(Rx) = (RR)\,x. \tag{1.1}$$

We will attempt to show why (1.1) is not a sufficiently strong assumption and why the following axioms, which are strengthenings of (1.1), are reasonable to assume:

AXIOM 1. *For all left ideals L_1 and L_2 of R and for every $x \in R$, $L_1(L_2x) = (L_1L_2)\,x$.*

AXIOM 2. *For every left ideal L of R and for all x and y in R, $x(Ly) = (xL)\,y$.*

AXIOM 3. *For every left ideal L of R and for all x and y in R, $L(xy) = (Lx)\,y$.*

One can readily observe that either of Axiom 2 or Axiom 3 implies Axiom 1.

The investigation of these axioms is carried out in Section 2 under the assumption that each ring has an identity element and satisfies either the descending chain condition or the ascending chain condition for left ideals.

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As the concept of left ideals for associative rings leads to the concept of R-modules, the concept of left ideals for rings under Axiom 3 leads in a natural way to the definition and investigation of $\bar{R}$-modules found in Section 3.

2. Rings

2.1. *Preliminary Concepts*

Definition 2.1. By a ring R we mean an ordered triple $\langle R, +, \cdot \rangle$ where $\langle R, + \rangle$ is an Abelian group and "$\cdot$" is a binary operation on R such that for all $a, b, c \in R$

(i) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

(ii) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

As usual, we adopt the convention of using juxtaposition instead of "$\cdot$".

Definition 2.2. If R is a ring we define $(\cdot, \cdot, \cdot) : R \times R \times R \to R$, called the associator of R, by $(x, y, z) = (xy)\, z - x(yz)$.

Definition 2.3. Suppose that R is a ring and $x \in R$. We define $x^1 = x$, $x^{n+1} = x^n x$ for $n \geqslant 1$. If R has identity 1 and $x \neq 0$, $x^0 = 1$.

We say that a non-zero element $x \in R$ is nilpotent with nilpotent exponent n if all possible products of x times itself n times yields 0 and some product of x times itself $(n - 1)$ times is not zero.

Definition 2.4. Suppose A and B are subsets of a ring R. We define

(i) $AB = \{ab : a \in A \text{ and } b \in B\}$,

(ii) $A \circ B = \{\sum c : c \in AB, \text{ finite sums}\}$,

(iii) $A^1 = A$, $A^{n+1} = A \circ A^n$ for $n \geqslant 1$, and

(iv) $\ell - \operatorname{ann}(A) = \{r \in R : rA = \{0\}\}$.

We say that A is nil if every element of A is nilpotent; we say that A is nilpotent with nilpotent exponent n if all possible products of any n elements of A yields 0 and, for $A \neq \{0\}$, some product of $(n - 1)$ many elements from A is not 0.

Definition 2.5. Suppose that R is a ring. We say that R is a left (right)-division ring if for every $0 \neq a$ and b in R, there is an x in R such that $xa = b$ $(ax = b)$. If R is both a left and right-division ring we call R a division ring. We call R a unique (left-, right-)division ring if R is a (left-, right-) division ring with no proper zero divisors.

NOTATION 2.6. We will use the notation ΣR_i for the internal sum, not necessarily direct, of the groups $\langle R_i, +\rangle$, $\Sigma \oplus R_i$ for the internal direct sum, and $\dot{+} R_i$ for the external direct sum.

Throughout this paper when we say that R satisfies D.C.C. (A.C.C.) we mean that R satisfies the descending (ascending) chain condition on its left ideals.

2.2. *Motivation*

We now list some theorems and examples, without proofs, for motivation:

THEOREM 2.7. *Any left-division ring satisfies Axiom 1.*

THEOREM 2.8 (Neumann, [3]). *Any algebra can be embedded in a (left-) division algebra with identity.*

THEOREM 2.9. *If D is a unique left-division ring, then D satisfies Axiom 3.*

THEOREM 2.10 (Neumann, [3]). *If A is an algebra with identity 1 which has no proper zero divisors, then A can be embedded in a unique (left-)division algebra with identity 1.*

EXAMPLE 2.11. Let $Z_p \subseteq F \subseteq K$ where K is the algebraic closure of the field F. Define the ring R by $R = Z_p \dot{+} F$ with the multiplication

$$(a, b) * (c, d) = (ac, ad + bc + b \circ d), \tag{2.1}$$

where "$\circ$" denotes a multiplication on F.

Case 1. If we take "$\circ$" defined by

$$b \circ d = bd^p,$$

then $\langle F, +, \circ\rangle$ satisfies Aixom 2 and Axiom 3, where R under "$*$" is the usual manner of adjoining an identity to the ring $\langle F, +, \circ\rangle$ but $\langle R, +, *\rangle$ does not even satisfy Axiom 1.

Case 2. Now take $F = K$ and define "$\circ$" by

$$b \circ d = b^p d^p.$$

Then $\langle R, +, *\rangle$ satisfies the following:

(i) R has an identity.

(ii) R has only one proper left ideal, $\{0\} \dot{+} K$.

(iii) R has no non-zero nil left ideals.

(iv) R is not a ring direct sum of any of its proper ideals.

(v) R satisfies Axiom 1 and Axiom 2.

(vi) R does not satisfy Axiom 3.

Remark. If R is a ring direct sum (product) of rings satisfying Axiom (i), then R satisfies Axiom (i), $1 \leqslant i \leqslant 3$.

Remark. It is possible to show, using the construction given in [*3*], that there exist unique left-division rings D with proper right ideals which do not satisfy Axiom 2 and which do not satisfy the dual to Axiom 1 where "left ideal" is replaced by "right ideal".

The following example gives some indication of the weakness of structure possible and the lack of duality in the left and right ideal structures for rings which satisfy (1.1) or the dual of Axiom 1.

EXAMPLE 2.12. Let R be the algebra over F, char $F \neq 2$, with basis e_1, x_1, e_2, and x_2 and the multiplication table:

	e_1	x_1	e_1	x_2
e_1	e_1	x_1	0	0
x_1	x_1	$-(\frac{1}{2}e_1)$	0	$-e_2$
e_2	0	0	e_2	x_2
x_2	0	e_1	x_2	$-(\frac{1}{2}e_2)$

(2.2)

The following hold for R:

(i) $e_1 + e_2$ is the identity for R.

(ii) For every $x \in R$, $R(Rx) = (RR)\,x$.

(iii) R has no proper right ideals.

(iv) For all right ideals R_1 and R_2 of R and every $x \in R$,

$$R_1(R_2x) = (R_1R_2)\,x.$$

(v) R is not a right or left-division algebra.

(vi) R does not satisfy Axiom 1, Axiom 2, or Axiom 3.

2.3. *Radical Theory*

PROPOSITION 2.13. *Suppose R satisfies Axiom 1. Then for every left ideal L of R and every $x \in R$, Lx is a left ideal of R. If R satisfies Axiom 3, then* ℓ-ann (T) *is a left ideal of R for every set $T \subseteq R$.*

Proof. Suppose L is a left ideal of R and $x \in R$. Clearly Lx is closed under addition. By Axiom 1, $R(Lx) = (RL)\,x \subset Lx$.

Suppose that R satisfies Axiom 3 and $T \subseteq R$. Clearly ℓ-ann (T) is an additive subgroup of R. Suppose $s \in R$, $r \in \ell$-ann (T), and $t \in T$. By Axiom 3, there is an $\bar{s} \in R$ such that $(sr)\,t = \bar{s}(rt) = 0$. Thus, $(sr)\,T = \{0\}$ and ℓ-ann (T) is a left ideal of R.

PROPOSITION 2.14. *Suppose R satisfies Axiom 1. For any positive integer n and any left ideal L of R, L^n is a left ideal of R.*

Proof. The statement is clear for $n = 1$. Suppose $n > 1$. By Definition 2.4 (iii) and Axiom 1,

$$RL^n = R(L \circ L^{n-1}) = (RL) \circ L^{n-1} \subset L \circ L^{n-1} = L^n.$$

PROPOSITION 2.15. *Suppose R satisfies Axiom 1. For any left ideal L of R and any positive integers n and m*

$$L^n L^m = L^{n+m}.$$

Proof. This is clear by Definition 2.4 (iii), Axiom 1, and Proposition 2.14.

PROPOSITION 2.16. *Suppose R satisfies Axiom 1. If L_1 and L_2 are nilpotent left ideals of R, then $L_1 + L_2$ is a nilpotent left ideal of R.*

Proof. Suppose that L_1 is a nilpotent left ideal of R with nilpotent exponent n and that L_2 is a nilpotent left ideal of R with nilpotent exponent m. Clearly, $L_1 + L_2$ is a left ideal of R.

By Proposition 2.15, to show that $L_1 + L_2$ is nilpotent, it suffices to show that any product

$$(\dots(((a_1 a_2)\, a_3)\, a_4) \dots)\, a_{m+n} \tag{2.3}$$

is zero, where each a_i belongs to L_1 or L_2.

Without loss of generality, we may assume that we have at least n elements from L_1 in the product (2.3). Since ba_i belongs to the same left ideal as a_i, we may also assume that $a_1 \in L_1$. Suppose we have written the product (2.3) in the form

$$(\dots(((b_1 b_2)\, b_3)\, b_4) \dots)\, b_t\,, \tag{2.4}$$

where $b_1 \in L_1{}^k$ and $b_i \in L_1$ or $b_i \in L_2$ for $2 \leqslant i \leqslant t$ with at least $n - k$ of the b_i, $i > 1$, belonging to L_1.

If $b_2 \in L_1$, then $b_1 b_2 \in L_1^{k+1}$ and

$$(\dots(((b_1 b_2) b_3) b_4) \dots) b_t = (\dots(((c_1 c_2) c_3) c_4) \dots) c_{t-1}$$

where $c_1 = b_1 b_2 \in L_1^{k+1}$, $c_i = b_{i+1}$ for $i > 1$ and at least $n - (k+1)$ of the c_i, $i > 1$, belong to L_1.

Suppose $b_2 \notin L_1$. Let $q > 2$ be the least integer such that $b_q \in L_1$. Axiom 1 now gives

$$\begin{aligned}(\dots(((b_1 b_2) b_3) b_4) \dots) b_t &= (\dots(((b_{11}(b_{21} b_3)) b_4) \dots) b_t \\ &= (\dots(((c_1 c_2) c_3) c_4) \dots) c_{t-1}, \end{aligned} \tag{2.5}$$

where $c_1 = b_{11} \in L_1{}^k$, $c_2 = b_{21} b_3$, $c_i = b_{i+1}$ for $i > 2$ and $c_{q-1} \in L_1$. Hence, by induction we may now assume that $b_3 \in L_1$. Then from (2.5) we have: $(\dots(((b_1 b_2) b_3) b_4) \dots) b_t = (\dots(((c_1 c_2) c_3) c_4) \dots) c_{t-1} = (\dots(((d_1 d_2) d_3) d_4) \dots) d_{t-2}$, where $d_1 = c_1 c_2 = c_1(b_{21} b_3) \in L_1^{k+1}$, $d_i = c_{i+1}$ for $i > 1$, and at least $n - (k+1)$ of the d_i, $i > 1$, belong to L_1.

By induction, any product of the form (2.3) with n many $a_i \in L_1$ belongs to $L_1{}^n$ or $L_1{}^n L_2$. But $L_1{}^n = L_1{}^n L_2 = \{0\}$.

PROPOSITION 2.17. *Suppose R satisfies Axiom 1 and L is a left ideal of R. Then $L \circ R$ is an ideal of R. If L is nilpotent, $L \circ R$ is nilpotent.*

Proof. Suppose $x \in L$ and $r, s \in R$. Then, by Axiom 1, there are $\bar{x} \in L$ and $\bar{r} \in R$ such that $(xr)s = \bar{x}(\bar{r}s) \in LR$. Also by Axiom 1, there is a $t \in R$ and a $y \in L$ such that $s(xr) = (ty)r \in LR$. Thus, $L \circ R$ is an ideal of R.

The remainder of the proof is similar to the reassociation argument found in the proof of Proposition 2.16.

DEFINITION 2.18. Suppose R satisfies Axiom 1 and A.C.C. By Proposition 2.16 and Proposition 2.17, R has a unique maximal nilpotent ideal N which we call the radical of R and denote by rad R. If rad $R = \{0\}$ we say that R is semi-simple.

PROPOSITION 2.19. *Suppose R satisfies Axiom 1,* A.C.C. *and has an identity. Then $\{0\} \neq R/\mathrm{rad}\, R$ and $R/\mathrm{rad}\, R$ is semi-simple.*

Proof. R has a maximal nilpotent left ideal L. By Proposition 2.17 $L \subset L \circ R$ where $L \circ R$ is a nilpotent ideal of R. Thus, $L \subset N$. If L_1/N is a nilpotent left ideal of R/N, then L_1 is a left ideal of R such that $L_1{}^n \subset N$, for some n. Then, $(L_1{}^n)^m = L_1^{nm} \subset N^m = \{0\}$, where m is the nilpotent exponent of N. Thus L_1 is nilpotent and $L_1 \subset N$.

We can obtain somewhat sharper results for radical theory under Axiom 3.

PROPOSITION 2.20. *Suppose R satisfies Axiom* 3, D.C.C. *and has an identity. Then a nil left ideal of R must be nilpotent.*

Proof. Suppose L is a nil but non-nilpotent left ideal of R. Without loss of generality we may assume that L is a minimal nil but non-nilpotent left ideal of R.

By Proposition 2.15, if L^2 is nilpotent, L is nilpotent. Hence $L^2 \subset L$ gives that $L^2 = L$.

Let $\mathcal{M} = \{M : M$ is a left ideal of R with $M \subseteq L$ and $LM \neq \{0\}\}$. $L \in \mathcal{M}$ so that $\mathcal{M} \neq \emptyset$. Let M be a minimal left ideal in $\mathcal{M}$.

$LM \neq \{0\}$ so that there is a $u \in M$ with $Lu \neq \{0\}$. By Proposition 2.13, Lu is a left ideal of R and $Lu \subset LM \subset M \subset L$. $L(Lu) = L^2u = Lu$, by Axiom 3 (or 1). Hence, $Lu = M$, by the minimality of M.

Hence there is an $e \in L$ such that $eu = u \neq 0$. Since $e \in L$ and L is nil, there is a positive integer n such that $e^n = 0$. But for $q \geqslant 1$,

$$e^q(e-1) = e^{q+1} - e^q \in \ell\text{-ann}\,(u),$$

by Proposition 2.13. Then $0 = e^n u = e^{n-1}u = \cdots = eu = u \neq 0$ is a contradiction. Thus every nil left ideal of R is nilpotent.

DEFINITION 2.21. If R satisfies Axiom 3, D.C.C. and has an identity element, we define the radical of R, denoted rad R, to be the sum of all its nilpotent left ideals. If rad $R = \{0\}$, we say that R is semi-simple.

PROPOSITION 2.22. *If R satisfies Axiom 3*, D.C.C., *and has an identity, then* rad R *is a nilpotent ideal of R and $R/$*rad *R is semi-simple.*

Proof. By Proposition 2.20, rad R is a nilpotent left ideal of R. By Proposition 2.17, (rad R) $\circ$ R is a nilpotent ideal of R so we have:

$$(\text{rad } R) \circ R \subset \text{rad } R \subset (\text{rad } R) \circ R$$

so that rad R is an ideal of R. Clearly $R/$rad R is semi-simple.

Note. If R satisfies Axiom 3, A.C.C., D.C.C., and has an identity, then Proposition 2.22 shows that the two definitions of rad R, Definition 2.19 and Definition 2.21, agree.

2.4. *Semi-simple Theory*

DEFINITION 2.23. Let R be a ring satisfying Axiom 1 and D.C.C. Suppose also that R has no non-zero nilpotent left ideals. Define $\mathcal{M}$ by

$$\mathcal{M} = \{L : L \text{ is a minimal left ideal of } R\}.$$

If $L_1, L_2 \in \mathcal{M}$ we write $L_1 \approx L_2$ if there is an $x \in L_2$ such that $L_2 = L_1 x$.

PROPOSITION 2.24. *Suppose we have R and $\mathcal{M}$ as in Definition 2.23. Then $\approx$ is an equivalence relation on $\mathcal{M}$. Furthermore, for $L_1, L_2 \in \mathcal{M}$, $L_1L_2 \neq \{0\}$ if and only if $L_1L_2 = L_2$, if and only if $L_1 \approx L_2$.*

The proof is straightforward, using the semi-simplicity of R and the minimality of the left ideals belonging to $\mathcal{M}$.

THEOREM 2.25. *Suppose R is a finite-dimensional, semi-simple algebra under Axiom 1. Then we may write $R = L \oplus M$, where $L = L_1 \oplus \cdots \oplus L_n$, a sum of minimal left ideals of R, and L is an ideal of R such that R contains no minimal left ideal not already contained in L. Hence, R has a composition series of left ideals.*

Proof. First suppose that I is a left ideal of R and L_1 is a minimal left ideal of R. Then either $L_1 \subset I$ or $L_1 \cap I = \{0\}$. Suppose $0 \neq x \in L_1 \cap I$. Then $x \in Rx \subset L_1$ and $Rx \subset I$. By the minimality of L_1, $L_1 = Rx \subset I$.

By the argument above, suppose we have $R = L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus M$ where the L_i are minimal left ideals of R and there are no minimal left ideals of R not contained in L.

Suppose that L is not an ideal of R. Then, since $L \circ R$ is an ideal of R, there is an $x \in R$ such that $L_ix \not\subset L$ for some i. L_ix must contain a minimal left ideal, say I. Now, $I = I^2 \subset I(L_ix) = (IL_i)x$ so that $I \approx L_i$ and $L_i \approx I$, by Proposition 2.24. But $\dim I \leqslant \dim (L_ix) \leqslant \dim L_i$ and, since for some $y \in L_i$, $L_i = Iy$, $\dim L_i \leqslant \dim (Iy) \leqslant \dim I$. Hence $I = L_ix$ and L_ix is a minimal left ideal. But $L_ix \not\subset L$ is a contradiction. Thus, L is an ideal of R.

Example 2.11, Case 2 shows that a semi-simple ring with identity under Axiom 2 need not be a direct sum of simple ideals. The difficulty here is that the sum of all minimal left ideals of R may not give all of R; see Theorem 2.25. However, we will see that this does not occur under Axiom 3.

Henceforth in Section 2.4 we will assume that R is a semi-simple ring with identity 1 under Axiom 3 and D.C.C.

PROPOSITION 2.26. *Suppose that L is a minimal left ideal of R. Then L contains a non-zero idempotent e. Furthermore, $L = Re$ and $R = Re \oplus R(1 - e)$, where the sum is additively direct.*

Proof. (a) Since R contains no nilpotent left ideals, $L^2 = L \neq \{0\}$. Then there is a $u \in L$ such that $Lu = L$. Hence there is an $e \in L$ such that $eu = u$. Then $e(e - 1) = e^2 - e \in L \cap \ell\text{-ann}(u)$. But $e \notin L \cap \ell\text{-ann}(u)$, so by Proposition 2.13, $L \cap \ell\text{-ann}(u) = \{0\}$ and $e^2 = e$. Clearly $Re = L$.

(b) $R = Re + R(1 - e)$ since $x = xe + x(1 - e)$.

Claim: $R(1 - e) = \{x \in R \mid xe = 0\}$.

If $x \in R(1-e)$, then $x = y(1-e)$ for some $y \in R$. Then,

$$xe = (y(1-e))\, e = \bar{y}((1-e)\, e) = 0.$$

Conversely, if $xe = 0$, then $x = x(1-e) \in R(1-e)$.

Now $Re \cap R(1-e) \subsetneq Re$ since $e \in Re$ and $e^2 = e \neq 0$. But $Re = L$ so that, by the minimality of L, $Re \cap R(1-e) = \{0\}$.

Proposition 2.27. *Every minimal left ideal of R is a direct summand of any containing left ideal. Also, we may write $R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$, where the Re_i are minimal left ideals and the e_i are non-zero idempotents.*

Proof. The proof is as in the associative case using Proposition 2.26.

Remark 2.28. We have

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n .$$

Write $1 = f_1 + f_2 + \cdots + f_n$, where $f_i \in L_i$. $f_i = f_i f_1 + f_i f_2 + \cdots + f_i f_n$ so that $f_i f_j = \delta_{ij} f_j$. Also, $L_i = Rf_i$ so we may replace every f_i by e_i and use $1 = e_1 + e_2 + \cdots + e_n$.

By Proposition 2.24, there are finitely many equivalence classes under $\approx$ in R, say k ($\leqslant n$) many. For each j with $1 \leqslant j \leqslant k$ pick a representative I_j for the k disjoint equivalence classes. Define $B_j = \sum_{L_i \approx I_j} L_i$. We may write $R = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, where each B_i is a simple component for R. Now, as in the associative case, we obtain:

Theorem 2.29. *If R is a semi-simple ring with identity under Axiom 3 and* D.C.C., *then R is a finite-ring direct sum of simple rings each satisfying Axiom 3 with descending chain condition and identity.*

2.5. *Simple Rings*

Theorem 2.30. *Suppose R is a finite-dimensional simple algebra with identity which satisfies Axiom 1. Then we may write $R = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, a direct sum of minimal left ideals L_i, where $1 = e_1 + e_2 + \cdots + e_n$ and $L_i = Re_i$. Furthermore, $L_i e_j = \delta_{ij} L_j$ and $\dim L_i = \dim L_j$ for $1 \leqslant i, j \leqslant n$.*

Proof. That we may write $R = L_1 \oplus \cdots \oplus L_n$ follows from Theorem 2.25 and the simplicity of R.

By the remarks preceding Theorem 2.29, $L_i \approx L_j$ for $1 \leqslant i, j \leqslant n$. This gives $\dim L_i \leqslant \dim L_j y \leqslant \dim L_j \leqslant \dim L_i x \leqslant \dim L_i$ so that $\dim L_i = \dim L_j$.

Write $1 = e_1 + e_2 + \cdots + e_n$ where $e_i \in L_i$. Then,

$$L_i = L_i(e_1 + \cdots + e_n) = \sum_{j=1}^{n} L_i e_j .$$

Thus, $L_i e_j = \delta_{ij} L_j$.

Henceforth in Section 2.5 we will assume that R is a simple ring with identity and satisfies one of the following:

(i) R is a finite-dimensional algebra satisfying Axiom 1.

(ii) R satisfies Axiom 3 and the descending chain condition.

By Proposition 2.27, Remark 2.28, and Theorem 2.30, we write $R = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ and $1 = e_1 + e_2 + \cdots + e_n$, $e_i \in L_i$.

LEMMA 2.31. *Let $a, x \in L_i$, $y \in L_j$. Then*

(a) $xy \neq 0$ *implies* $L_i y = L_j$.

(b) $ay = xy$ *implies* $a = x$ *or* $xy = 0$.

(c) $L_i e_j = \delta_{ij} L_j$.

(d) $L_i L_j = L_j$.

Proof. We have already shown (c) and (d) and only list them here for easy reference.

(a) $xy \neq 0$ gives $\{0\} \neq L_i y \subset L_j$. Then by the minimality of L_j, $L_i y = L_j$.

(b) Suppose $ay = xy \neq 0$. Then $L_i y = L_j$. For R satisfying (i), $\dim L_i = \dim L_j$. Since the mapping $R(y) : L_i \to L_j$ by $\ell_i \to \ell_i y$ is onto L_j, it is also one-to-one. Hence, $(a - x) y = 0$ gives $a = x$. For R satisfying (ii), $L_i \cap \ell\text{-ann}(y) = \{0\}$ since $x \in L_i$ is such that $xy \neq 0$ and L_i is a minimal left ideal. Thus $(a - x) y = 0$ gives $a = x$.

In the next lemma we wish to show certain associations of elements in R with idempotents e_i.

Because this lemma will also be useful for results in Section 3, we will specifically give the proofs for parts (b) and (c) for Axiom 3 where it will be helpful to consider s as belonging to a group M operated on by R rather than being an element of R.

We further note that, because of part (c) of the following lemma, Axiom 2 appears to be stronger than Axiom 1 even in the finite-dimensional case.

LEMMA 2.32. *For any $r, s \in R$, $1 \leqslant i \leqslant n$ we have*

(a) $(r, s, e_i) = 0$.

(b) $(r, e_i, s) = 0$.

If R satisfies Axiom 2 or 3 we also have

(c) $(e_i, r, s) = 0$.

Proof. By Lemma 2.31 part (c) we have $(re_i)\, e_j = 0$ for $i \neq j$. Now

$$(re_i) = (re_i) \cdot 1 = (re_i) \left(\sum_{j=1}^{n} e_j\right) = (re_i)\, e_i .$$

Next we wish to show that $L_i(e_j x) = \{0\}$ for $i \neq j$. For Axiom 3, Lemma 2.31 part (c) makes it clear that $L_i(e_j x) = \{0\}$ for $i \neq j$.

Suppose $L_i(e_j x) \neq \{0\}$ for some $i \neq j$. Let

$$x = xe_1 + xe_2 + \cdots + xe_n = x_1 + x_2 + \cdots + x_n .$$

First we wish to show that $L_i(e_j x_k) = \{0\}$ for $k \neq j$.

Suppose that $L_i(e_j x_k) \neq \{0\}$ for $k \neq j$. Then $L_i(e_j x_k) = L_k$. Now consider the left ideal $L_j(x_k + e_j)$.

$$\begin{aligned}\{0\} \neq L_k &= L_i(e_j x_k) = L_i(e_j x_k + e_j) = L_i(e_j(x_k + e_j)) \subseteq L_i(L_j(x_k + e_j)) \\ &= (L_i L_j)\,(x_k + e_j) = L_j(x_k + e_j).\end{aligned}$$

Since $e_j x_k \neq 0$, $L_j x_k = L_k$ and $L_j e_j = L_j$ (R is finite-dimensional under Axiom 1). Thus, $L_j \oplus L_k \subseteq L_j(x_k + e_j)$, which contradicts the dimension of $L_j(x_k + e_j)$.

Thus, $e_i(e_j x_k) = 0$ for $i \neq j$, $j \neq k$. Now we have

$$e_j x = \sum_{q=1}^{n} e_q(e_j x) = \sum_{q \neq j} e_q(e_j x_j) + e_j(e_j x). \tag{2.6}$$

Multiplying (2.6) on the left by e_j gives $e_j(e_j x) = e_j(e_j(e_j x))$. This also gives $e_i(e_i y) = e_i(e_i(e_i y))$ and replacing y by $e_j x$,

$$e_i(e_i(e_j x)) = e_i(e_i(e_i(e_j x))). \tag{2.7}$$

Left multiplication of (2.6) by e_i gives $e_i(e_j x) = e_i(e_i(e_j x)) + e_i(e_j(e_j x))$. Multiply this equation on the left by e_i to obtain

$$e_i(e_i(e_j x)) = e_i(e_i(e_i(e_j x))) + e_i(e_i(e_j(e_j x)))$$

which reduces, by (2.7), to $e_i(e_i(e_j(e_j x))) = 0$. But $e_k(e_i(e_j(e_j x))) = 0$ for $i \neq k$, since $e_i(e_j(e_j x_q)) = 0$ for $q \neq j$ and $e_k(e_i(e_j x_j)) = 0$. Hence for $i \neq j$,

$$e_i(e_j(e_j x)) = \sum_{k=1}^{n} e_k(e_i(e_j(e_j x))) = 0. \tag{2.8}$$

Now $e_i(e_j x) = e_i(e_i(e_j x))$, by multiplying (2.6) on the left by e_i. Since we already have $e_i(e_ix) = e_i(e_i(e_ix))$, we sum the last equation over j to obtain $\sum_{j=1}^{n} e_i(e_jx) = e_ix = e_i(e_ix)$. Hence, $e_jx = e_j(e_jx)$. Now, by (2.8), $e_i(e_jx) = e_i(e_j(e_jx)) = 0$, for $i \neq j$. Thus, by the finite-dimensionality under (ii) we have $L_i(e_jx) = \{0\}$, for $i \neq j$, and $e_j(e_jx) = e_jx$.

So far we have

$$(re_i)\, e_j = \delta_{ij}(re_j), \tag{2.9}$$

$$e_i(e_jr) = \delta_{ij}(e_jr), \tag{2.10}$$

and

$$(se_i)\,(e_jr) = 0, \qquad \text{for} \qquad i \neq j. \tag{2.11}$$

Claim: $(re_i)\,(se_j) = ((re_i)\, s)\, e_j$, for all i, j.

$$((re_i)\, s)\, e_j = \left((re_i)\left(\sum_{t=1}^{n} se_t\right)\right) e_j = ((re_i)\,(se_j))\, e_j + \left((re_i)\left(\sum_{t\neq j} se_t\right)\right) e_j .$$

But $((re_i)\,(se_t))\, e_j \in L_te_j = \{0\}$ for $t \neq j$ so that $((re_i)\, s)\, e_j = ((re_i)\,(se_j))\, e_j$. This gives $(((re_i)\, s)\, e_j)\, e_j = (((re_i)\,(se_j))\, e_j)\, e_j = ((re_i)\,(se_j))\, e_j$ so that $((re_i)\, s)\, e_j - (re_i)\,(se_j))$ belongs to $L_j \cap \ell\text{-ann}\,(e_j)$. But $L_j \cap \ell\text{-ann}\,(e_j) = \{0\}$, so that our claim follows.

Now,

$$(rs)\, e_j = \left(\sum_{i=1}^{n} (re_i)\, s\right) e_j = \sum_{i=1}^{n} ((re_i)\, s)\, e_j = \sum_{i=1}^{n} (re_i)\,(se_j) = r(se_j)$$

and (a) has been established.

By (2.11),

$$r(e_is) = \left(\sum_{j=1}^{n} (re_j)\right)(e_is) = \sum_{j=1}^{n} (re_j)\,(e_is) = (re_i)\,(e_is) = (re_i)\left(\sum_{j=1}^{n} (e_js)\right) = (re_i)\, s,$$

and (b) has been established.

Now we wish to show (c) using either Axiom 2 or Axiom 3.

Claim: $e_t((e_ir)\, s) = 0$ for $i \neq t$. Under Axiom 2, $e_t((e_ir)\, s) = e_t(e_i(\bar{r}s)) = 0$, by (2.10). Under Axiom 3, $e_t((e_ir)\, s) = ((\ell e_t)\,(e_ir))\, s = 0$, by (2.11).

Now,

$$e_i(rs) = e_i\left(\left(\sum_{t=1}^{n} e_tr\right) s\right) = \sum_{t=1}^{n} (e_i((e_tr)\, s)) = e_i((e_ir)\, s),$$

by the claim above. Hence,

$$(e_ir)\, s = \sum_{t=1}^{n} e_t((e_ir)\, s) = e_i((e_ir)\, s) = e_i(rs).$$

This establishes (c) under either Axiom 2 or Axiom 3.

DEFINITION 2.33. By Lemma 2.32, e_iRe_j is well-defined and we let $R_{ij} = e_iRe_j$.

Henceforth in Section 2.5, we will make further assumptions on the ring $R : R$ either satisfies part (c) of Lemma 2.32 under Axiom 1 or R satisfies Axiom 2 or Axiom 3.

LEMMA 2.34. $R_{ij}R_{tk} = \delta_{jt}R_{ik}$.

Proof. The proof of $R_{ij}R_{tk} = \{0\}$, for $j \neq t$, follows by Lemma 2.32 or assumption.

$$(e_iRe_j)(e_jRe_k) = e_i((Re_j)(e_jRe_k)) = e_i((Re_j)(Re_k)) = e_i(Re_k) = e_iRe_k .$$

THEOREM 2.35. *If we write $R = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ where $n \geqslant 2$, then R is associative and therefore a full-matrix algebra of unique finite-dimension over a skew field (Artinian Structure Theorem).*

Proof. By Definition 2.33 and Lemma 2.34, we write $R = \sum_{1 \leqslant i,j \leqslant n} \oplus R_{ij}$, where $R_{ij}R_{tk} = \delta_{jt}R_{ik}$.

Let $A = (a_{rt})$, $\bar{A} = (\bar{a}_{rt})$, $B = (b_{tj})$, $\bar{B} = (\bar{b}_{tj})$, and $C = (c_{jk})$. Let us consider

$$\bar{A}(\bar{B}C) = \left(\sum_{t=1}^{n} \sum_{j=1}^{n} \bar{a}_{rt}(\bar{b}_{tj}c_{jk})\right) \tag{2.12}$$

and

$$(AB)\,C = \left(\sum_{t=1}^{n} \sum_{j=1}^{n} (a_{rt}b_{tj})\,c_{jk}\right). \tag{2.13}$$

We wish to consider (2.12) and (2.13) where A and $\bar{A}$ are chosen from the left ideal of R determined by $t = i$ and where C is such that all $c_{mn} = 0$, except possibly c_{ii}, c_{ij}, c_{iq}, c_{jq} where $i \neq j$ and $i \neq q$. Fix $s \neq i$.

Now, given A and B we wish to solve for $\bar{A}$ and $\bar{B}$ in

$$\sum_{j=1}^{n} \bar{a}_{ri}(\bar{b}_{ij}c_{jk}) = \sum_{j=1}^{n} (a_{ri}b_{ij})\,c_{jk} . \tag{2.14}$$

We consider the cases $r = s$, $r = i$ in (2.14). Equation (2.14) implies that we must solve for every $\bar{a}$ and $\bar{b}$ in the following equations:

$$\bar{a}_{ii}(\bar{b}_{ii}c_{ii}) + \bar{a}_{ii}(\bar{b}_{ij}c_{ji}) = (a_{ii}b_{ii})\,c_{ii} + (a_{ii}b_{ij})\,c_{ji} , \tag{2.15}$$

$$\bar{a}_{ii}(\bar{b}_{ii}c_{iq}) + \bar{a}_{ii}(\bar{b}_{ij}c_{iq}) = (a_{ii}b_{ii})\,c_{iq} + (a_{ii}b_{ij})\,c_{jq} , \tag{2.16}$$

$$\bar{a}_{si}(\bar{b}_{ii}c_{ii}) + \bar{a}_{si}(\bar{b}_{ij}c_{ji}) = (a_{si}b_{ii})\,c_{ii} + (a_{si}b_{ij})\,c_{ji} , \tag{2.17}$$

$$\bar{a}_{si}(\bar{b}_{ii}c_{iq}) + \bar{a}_{si}(\bar{b}_{ij}c_{jq}) = (a_{si}b_{ii})\,c_{iq} + (a_{si}b_{ij})\,c_{jq} . \tag{2.18}$$

Given c_{ii}, c_{ji} non-zero and one of b_{ii}, b_{ij}, we can solve for the other b_{ii}, b_{ij} so that

$$b_{ii}c_{ii} + b_{ij}c_{ji} = 0. \tag{2.19}$$

Suppose we have (2.19). Now we take c_{iq}, c_{jq} (one may be chosen to be zero) such that $b_{ii}c_{iq} + b_{ij}c_{jq} \neq 0$. Let $a_{ii} = e_i$. From (2.16) and Lemma 2.32 part (c) we have that $\bar{a}_{ii} \neq 0$. By (2.15) we have that $\bar{b}_{ii}c_{ii} + \bar{b}_{ij}c_{ji} = 0$. Hence, from (2.17) we learn

$$(a_{si}b_{ii})\, c_{ii} + (a_{si}b_{ij})\, c_{ji} = 0 \qquad \text{for all} \qquad a_{si} \in R_{si}. \tag{2.20}$$

First let b_{ij}, c_{ji} be non-zero and $c_{ii} = e_i$. Choose b_{ii} so that (2.19) holds. From (2.20), for any $a_{si} \in R_{si}$, we have

$$a_{si}(b_{ii}c_{ii} + b_{ij}c_{ji}) = 0 = (a_{si}b_{ii})\, c_{ii} + (a_{si}b_{ij})\, c_{ji}.$$

But $a_{si}(b_{ii}e_i) = (a_{si}b_{ii})\, e_i$ by Lemma 2.32, so we obtain

$$a_{si}(b_{ij}c_{ji}) = (a_{si}b_{ij})\, c_{ji}, \tag{2.21}$$

for any $a_{si} \in R_{si}$, $b_{ij} \in R_{ij}$, $c_{ji} \in R_{ji}$, $s \neq i, j \neq i$.

The remainder of the associations are proven in a similar manner proceeding with subscripts

$(si)\,(ii)\,(ii)$,	$s \neq i$		
$(si)\,(ii)\,(ij)$,	$s \neq i$,	$j \neq i$	
$(si)\,(ij)\,(jq)$,	$s \neq i$,	$j \neq i$,	$q \neq i$
$(ii)\,(ii)\,(ij)$,	$j \neq i$		
$(ii)\,(ij)\,(jq)$,	$j \neq i$,	$q \neq i$	
$(ii)\,(ij)\,(ji)$,	$j \neq i$		
$(ii)\,(ii)\,(ii)$.			

This gives the associativity of R.

Now we have the following theorems:

THEOREM 2.36. *If R is a finite-dimensional simple algebra with identity which satisfies either of the following:*

(i) *R satisfies Axiom 1 and Lemma 2.32 part (c), or*

(ii) *R satisfies Axiom 2;*

then R is either a unique-division algebra with identity or (by Wedderburn Structure Theorem) a full-matrix algebra of unique finite-dimension over a skew field.

Proof. Theorem 2.35 shows that R is associative if R has a proper left ideal.

Suppose R has no proper left ideals and $0 \neq x \in R$. Then $0 \neq x = 1 \cdot x \in Rx$. Hence, $\{0\} \neq Rx = R$. Thus R is a left-division ring. Since R is a finite-dimensional algebra, R is a unique-division algebra.

THEOREM 2.37. *If R is a simple ring with identity which satisfies* D.C.C. *and Axiom 3, then R is either a unique left-division algebra with identity or (by the Artinian Structure Theorem) a full-matrix algebra of unique finite-dimension over a skew field.*

Proof. Theorem 2.35 proves that R is associative if R has a proper left ideal.

Suppose R has no proper left ideals and $0 \neq x \in R$. Then $0 \neq x = 1 \cdot x \in Rx$. Hence, $\{0\} \neq Rx = R$. Thus R is a left-division ring. If $xy = 0$, where $x \neq 0$, $y \neq 0$, then $\{0\} = R(xy) \neq R = Ry = (Rx)y$, and Axiom 3 would not be satisfied. Since every simple ring is an algebra over its centralizer [5, p. 14], if R has no proper left ideals, R is a unique left-division algebra.

3. MODULES

3.1 *Preliminary Concepts*

Throughout Section 3, R will denote a ring with identity 1.

DEFINITION 3.1. Let M be an Abelian group under "$+$". We call M a uniform $\bar{R}$-module under "$\cdot$" if $\cdot : R \times M \to M$ is a function which satisfies

(i) $1 \cdot m = m$ for all $m \in M$,

(ii) $(a + b) \cdot m = (a \cdot m) + (b \cdot m)$ for all $a, b \in R$, $m \in M$,

(iii) $a \cdot (m + n) = (a \cdot m) + (a \cdot n)$ for all $a \in R$, $m, n \in M$,

(iv) for all a, $b \in R$, $m \in M$, there is an $\bar{a} \in Ra$ such that $(ab) \cdot m = \bar{a} \cdot (b \cdot m)$, and

(v) for all $\bar{a}$, $b \in R$, $m \in M$, there is an $a \in R\bar{a}$ such that $(ab) \cdot m = \bar{a} \cdot (b \cdot m)$.

Again, juxtaposition will be used instead of "$\cdot$" when ring elements and module elements are easily distinguished.

DEFINITION 3.2. Let M be an Abelian group under "$+$". We call M an $\bar{R}$-module if there exist subgroups M_i of M for $i \in \Lambda$ such that each M_i is a uniform $\bar{R}$-module and $M = \sum_{i \in \Lambda} M_i$.

DEFINITION 3.3. Let M and N be two $\bar{R}$-modules and let $T : M \to N$ be a function from M to N. We call T an R-homomorphism of M to N if

(i) $T(m + n) = T(m) + T(n)$, for all $m, n \in M$,

(ii) $T(rm) = rT(m)$, for all $r \in R$, $m \in M$.

We say that T is a uniform $\bar{R}$-homomorphism if T satisfies (i) above as well as the following:

(iii) For every $r \in R$, $m \in M$, there is an $\bar{r} \in Rr$ such that $T(rm) = \bar{r}T(m)$, and

(iv) For every $\bar{r} \in R$, $m \in M$, there is an $r \in R\bar{r}$ such that $T(rm) = \bar{r}T(m)$.

It is clear how we would extend these definitions to those of (uniform) $R(\bar{R})$-monomorphisms, epimorphisms, isomorphisms, etc.

LEMMA 3.4. *Let M and N be $\bar{R}$-modules and $T : M \to N$ an onto mapping.*

(i) *If T is an R-isomorphism, so is T^{-1}.*

(ii) *If T is a uniform $\bar{R}$-isomorphism, so is T^{-1}.*

Proof. Clearly T^{-1} is additive. (i) and (ii) follow from

$$T^{-1}(\bar{r}n) = T^{-1}(\bar{r}T(m)) = T^{-1}(T(rm)) = rm = rT^{-1}(n).$$

LEMMA 3.5. *Let M and N be (uniform) $\bar{R}$-modules with $N \subseteq M$. Consider $\nu : M \to M/N$ defined by $\nu(m) = m + N$. Using the usual definitions of addition and module multiplication on M/N, M/N is an (uniform) $\bar{R}$-module and ν is an R-homomorphism of M onto M/N. ν is called the natural (or canonical) epimorphism.*

Proof. Identical to the associative case.

LEMMA 3.6. *Let M and N be uniform $\bar{R}$-modules and $T : M \to N$ an R-homomorphism (uniform $\bar{R}$-homomorphism) of M onto N. Let* $\ker T = T^{-1}(\{0\})$. *Then* $\ker T$ *is a uniform $\bar{R}$-submodule of M and $M/\ker T$ is R-isomorphic (uniformly $\bar{R}$-isomorphic) to N.*

Proof. Suppose $r \in R$, $x, y \in \ker T$.

$$T(r(x + y) = \bar{r}T(x + y) = \bar{r}(T(x) + T(y)) = 0.$$

Thus $r(x + y) \in \ker T$. Hence $\ker T$ is a uniform $\bar{R}$-submodule of M.

Let $\bar{T} : M/\ker T \to N$ by $\bar{T}(\nu(m)) = T(m)$, where ν is the natural epimorphism of M onto $M/\ker T$. As in R-module theory, $\bar{T}$ is well-defined. By Lemma 3.5, $\bar{T}(r(\nu(m))) = \bar{T}(\nu(rm)) = T(rm) = \bar{r}T(m) = \bar{r}\bar{T}(\nu(m))$, where

$\tilde{r} = r$ if T is an R-homomorphism. Thus $\tilde{T}$ is an R-isomorphism (uniform $\bar{R}$-isomorphism) of $M/\ker T$ onto N.

LEMMA 3.7. *Let N and L be $\bar{R}$-submodules of the $\bar{R}$-module M. Then $(L + N)/L$ is R-isomorphic to $N/L \cap N$.*

Proof. Identical to the associative case.

LEMMA 3.8. *Let N be a submodule of the $\bar{R}$-module M. There is a one-to-one inclusion preserving correspondence between the submodules of M which contain N and the submodules of M/N. Furthermore, if $N \subset L \subset M$, then M/L is R-isomorphic to $(M/N)/(L/N)$.*

Proof. Identical to the associative case.

LEMMA 3.9. *Let M be an $\bar{R}$-module with submodules M_i, $1 \leqslant i \leqslant k$, such that $M = \sum_{i=1}^{k} \oplus M_i$. For each i let N_i be a submodule of M_i and set $N = N_1 + N_2 + \cdots + N_k$. Then, $N = \sum_{i=1}^{k} \oplus N_i$ and M/N is R-isomorphic to $M_1/N_1 \dotplus \cdots \dotplus M_k/N_k$.*

Proof. Identical to the associative case.

DEFINITION 3.10. Let M be an $\bar{R}$-module and V a subset of M. We define ℓ-ann (V) by

$$\ell\text{-ann}\,(V) = \{r \in R \mid rV = \{0\}\}.$$

LEMMA 3.11. *Suppose that M is a uniform $\bar{R}$-module and that V is a subset of M. Then ℓ-ann (V) is a left ideal of R.*

Proof. Suppose s, $t \in \ell$-ann (V), $r \in R$, and $v \in V$.

$$(r(s + t)) \cdot v = \bar{r} \cdot ((s + t) \cdot v) = \bar{r} \cdot ((s \cdot v) + (t \cdot v)) = 0.$$

Thus $r(s + t) \in \ell$-ann (V) and ℓ-ann (V) is a left ideal of R.

The following technical lemma will be useful later:

LEMMA 3.12. *Suppose that M is a non-zero uniform $\bar{R}$-module and that G and H are additive subgroups of R. Suppose we also have*

(i) *for $a \in R$, $m \in M$, $a \cdot m = 0$ implies $a = 0$ or $m = 0$,*

(ii) *for $a \in G$, $b \in H$, $ab = 0$ implies $a = 0$ or $b = 0$.*

Fix $0 \neq m \in M$. Suppose, given $a \in G$, $b \in H$, the solution for $\bar{a} \in R$, such that $a \cdot (b \cdot m) = (\bar{a}b) \cdot m$, lies in G. Define $T_G : H \times G \to G$ by $T_G(b, a) = \bar{a}$ where $a \cdot (b \cdot m) = (\bar{a}b) \cdot m$. Then T_G is a well-defined function which satisfies

$$T_G(b, a + c) = T_G(b, a) + T_G(b, c) \tag{3.1}$$

and

$$T_G(b + d, a)(b + d) = T_G(b, a)\, b + T_G(d, a)\, d. \tag{3.2}$$

Hence, if we define $\circ : G \times H \to R$ *by*

$$a \circ b = T_G(b, a)\, b, \tag{3.3}$$

Eqs. (3.1) and (3.2) give the distributivity of "$\circ$".

Proof. Let G, H, $m \in M$ have the properties mentioned above.

Since $a \cdot m = 0$ and $m \neq 0$ implies that $a = 0$ and $ab = 0$ implies that $a = 0$ or $b = 0$, the solution for $\bar{a}$, given $a \in G$ and $b \in H$, in

$$a \cdot (b \cdot m) = (\bar{a}b) \cdot m$$

is unique so that T_G is a well-defined function.

Let $b, d \in H$, $a, c \in G$. Then

$$a \cdot (b \cdot m) + c \cdot (b \cdot m) = (a + c) \cdot (b \cdot m) = (T_G(b, a + c)\, b) \cdot m$$

and

$$\begin{aligned} a \cdot (b \cdot m) + c \cdot (b \cdot m) &= (T_G(b, a)\, b) \cdot m + (T_G(b, c)\, b) \cdot m \\ &= [(T_G(b, a) + T_G(b, c))\, b] \cdot m. \end{aligned}$$

Then (i) and (ii) now give

$$T_G(b, a + c)\, b = [T_G(b, a) + T_G(b, c)]\, b$$

and

$$T_G(b, a + c) = T_G(b, a) + T_G(b, c).$$

Thus we have (3.1).

Also,

$$\begin{aligned} a \cdot (b \cdot m) + a \cdot (d \cdot m) &= (T_G(b, a)\, b) \cdot m + (T_G(d, a)\, d) \cdot m \\ &= [T_G(b, a)\, b + T_G(d, a)\, d] \cdot m \end{aligned}$$

and

$$a \cdot (b \cdot m) + a \cdot (d \cdot m) = a \cdot ((b + d) \cdot m) = [T_G(b + d, a)(b + d)] \cdot m$$

so that

$$T_G(b, a)\, b + T_G(d, a)\, d = T_G(b + d, a)(b + d),$$

which is (3.2).

3.2. *Classification of Uniform $\bar{D}$-Modules, D a Unique Left-Division Ring with Identity*

Throughout 3.2, D will denote a unique left-division ring $\langle D, +, \cdot \rangle$ with identity 1. We will use the notation $|\Lambda|$ to denote the cardinality of a set Λ.

DEFINITION 3.13. Let M be a $\bar{D}$-module. The elements $x_1, x_2, \ldots, x_n$ in M are said to be dependent if there are $a_i \in D$, not all $a_i = 0$, such that $\sum_{i=1}^{n} a_i x_i = 0$. If no such a_i exist, then the x_i, $1 \leqslant i \leqslant n$, are called independent. $\{x_i\}_{i \in \Lambda}$, a subset of M, is said to be an independent set if, for every finite subset Γ of Λ, $\{x_i\}_{i \in \Gamma}$ is independent. Otherwise $\{x_i\}_{i \in \Lambda}$ is called a dependent set.

Remark 3.14. Suppose M is a uniform $\bar{D}$-module and $\{x_i\}_{i \in \Lambda}$ is an independent subset of M but, for $y \in M$, $\{x_i, y\}_{i \in \Lambda}$ is a dependent set. Then, for some finite subset Γ of Λ, there are a, $a_t \in D$, $t \in \Gamma$ such that $a \neq 0$ and $ay + \sum_{t \in \Gamma} a_t x_t = 0$. Then there is a $b \in D$ such that $b(ay) = (a^{-1}a)\,y = y$ and $y = \sum_{t \in \Gamma} (\bar{b}_t a_t)\, x_t$. Similarly, if $0 \neq x \in M$, then $ax = 0$, $a \in D$ implies that $a = 0$.

DEFINITION 3.15. Suppose M is a uniform $\bar{D}$-module and $\{x_i\}_{i \in \Lambda} \subseteq M$. $\{x_i\}_{i \in \Lambda}$ is an independent generating set if $\{x_i\}_{i \in \Lambda}$ is an independent set and for any $y \in M$, $\{x_i, y\}_{i \in \Lambda}$ is dependent. We define the dimension of M to be the minimum cardinality of independent generating sets. (Independent generating sets are guaranteed by Zorn's lemma and Remark 3.14).

THEOREM 3.16. *Suppose that $\langle D, +, \circ \rangle$ is a unique left-division ring with identity 1 where 1 is also the identity under "$\cdot$" for D. Then $\langle D, + \rangle$ is a uniform $\bar{D}$-module of dimension 1 using "$\circ$" as the module operation. Conversely, every uniform $\bar{D}$-module of dimension 1 is D-isomorphic to $\langle D, + \rangle$ for some module operation "$\circ$" as above.*

Proof. The first part above is clear.

Suppose M is a uniform $\bar{D}$-module of dimension 1 and let $\{x\}$ be an independent generating set for M. Define $T_D : D \times D \to D$ as in Lemma 3.12 using $x = m \in M$, $G = H = D$. Then, by Remark 3.14 and the fact that D is a unique left-division ring, "$\circ$", as defined in Lemma 3.12, is a binary distributive operation on D.

Since M is unital by definition, $1 \circ b = b = b \circ 1$ and 1 is the identity for D under "$\circ$". By the uniqueness of solutions in (iv) and (v) of Definition 3.1, $T_D(b, \cdot)$ is one-to-one and onto for $0 \neq b \in D$. Thus, since $\langle D, +, \cdot \rangle$ is a unique left-division ring, $\langle D, +, \circ \rangle$ is a unique left-division ring with identity 1. Define $\sigma : M \to D$ by $\sigma(ax) = a$. σ is a D-isomorphism of M onto $\langle D, + \rangle$ under "$\circ$" since

$$\sigma(a \cdot (b \cdot x)) = \sigma((a \circ b) \cdot x) = a \circ b = a \circ \sigma(b \cdot x).$$

Henceforth in Section 3.2 let M be a uniform $\bar{D}$-module of dimension $|\Lambda| \geqslant 2$ with $\{x_i\}_{i \in \Lambda}$ an independent generating set for M.

LEMMA 3.17. *Suppose $\bar{a}$, a, c are non-zero elements of D such that for some x_i with $i \in \Lambda$, $a \cdot (c \cdot x_i) = (\bar{a}c) \cdot x_i$. Then, if $a \cdot (c \cdot x_i) = (bc) \cdot x_i$, $b = \bar{a}$ and for every $y \in M$, $a \cdot (c \cdot y) = (\bar{a}c) \cdot y$.*

Proof. Write $x_i = x$ and suppose $a(cx) = (bc)\, x$. Then, $(\bar{a}c)\, x = (bc)\, x$ and $(\bar{a}c - bc)\, x = 0$. Thus, by Remark 3.14, $\bar{a}c - bc = 0$. Since $c \neq 0$, $\bar{a} = b$.

Let $y \in M$. We cite two cases.

Case 1. Suppose y is independent of x.

$$(\bar{a}c)\,(x + y) = (\bar{a}c)\, x + (\bar{a}c)\, y = a(cx) + a'(cy)$$

and

$$(\bar{a}c)\,(x + y) = a''(c(x + y)) = a''(cx) + a''(cy).$$

Since x and y are independent, $c \neq 0$, cx and cy are independent. Then $a = a'' = a'$ and $(\bar{a}c)\, y = a(cy)$.

Case 2. By Case 1, $a(cx_j) = (\bar{a}c)\, x_j$ for $j \neq i$. Again by Case 1, for $d \neq 0$, dx is independent of x_j, $j \neq i$, and $a(c(dx)) = (\bar{a}c)\,(dx)$ and the lemma follows.

For $c \neq 0$ in D, consider $T_D : D \times D \to D$ given in Lemma 3.12 and "$\circ$" given by $a \circ b = T_D(b, a)\, b$, as in Lemma 3.12. Define $M' = \dot{+}_{i \in \Lambda} D$. We denote the ith component of an element $m \in M'$ by m_i. Now define $\circ : D \times M' \to M'$ by using "$\circ$" above in components, i.e.:

$$(a \circ m)_i = a \circ m_i, \qquad i \in \Lambda. \tag{3.4}$$

Define the mapping $\sigma : M \to M'$ by

$$\left[\sigma\left(\sum_{i \in \Lambda} a_i x_i\right)\right]_j = a_j, \qquad j \in \Lambda. \tag{3.5}$$

σ is clearly additive, one-to-one, and onto M'.

Next we wish to show that σ is a D-isomorphism of M onto M' under "$\circ$".

Let $x = \sum_{i \in \Lambda} c_i x_i \in M$. By the definition of σ, for $a \in D$,

$$\left[\sigma\left(a\left(\sum_{i \in \Lambda} c_i x_i\right)\right)\right]_j = \left[\sigma\left(\sum_{i \in \Lambda} (T_D(c_i, a)\, c_i)\, x_i\right)\right]_j = T_D(c_j, a)\, c_j = a \circ c_j$$
$$= a \circ (\sigma(x))_j = (a \circ \sigma(x))_j.$$

Hence $\sigma(a \cdot x) = a \circ \sigma(x)$.

Again, since M is unital, $c \cdot (1 \cdot x) = c \cdot x = 1 \cdot (c \cdot x)$ so that $1 \circ c = c = c \circ 1$, $c \in D$ and 1 is the identity for D under "$\circ$". Hence, M' under "$\circ$" is a uniform $\bar{D}$-module D-isomorphic to M.

The following two equations are consequences of Lemma 3.17:

$$(ab) \cdot (c \cdot x) = T_D^{-1}(b, a) \cdot (b \cdot (c \cdot x)) = T_D^{-1}(b, a) \cdot ((T_D(c, b)\, c) \cdot x)$$
$$= [(T_D(T_D(c, b)\, c,\ T_D^{-1}(b, a)))\,(T_D(c, b)\, c)] \cdot x, \tag{3.6}$$
$$(ab) \cdot (c \cdot x) = (T_D(c, ab)\, c) \cdot x, \tag{3.7}$$

Equation (3.6) follows from Lemma 3.17 in that Lemma 3.17 shows that $T_D^{-1}(b, a)$, in (3.6), is independent of c. By Eqs. (3.6) and (3.7), we have for all $a, b, c \in D$,

$$T_D(c, ab)\, c = T_D(T_D(c, b)\, c,\ T_D^{-1}(b, a))\,(T_D(c, b)\, c). \tag{3.8}$$

Now we have

$$(a \circ b) \circ c = T_D(c, a \circ b)\, c = T_D(c, T_D(b, a)\, b)\, c \tag{3.9}$$

and

$$a \circ (b \circ c) = T_D(b \circ c, a)\,[b \circ c] = T_D(T_D(c, b)\, c, a)\,[T_D(c, b)\, c]. \tag{3.10}$$

In (3.8) replace $T_D^{-1}(b, a)$ by a and hence a by $T_D(b, a)$ and obtain

$$T_D(c, T_D(b, a)\, b)\, c = T_D(T_D(c, b)\, c, a)\,(T_D(c, b)\, c). \tag{3.11}$$

Equations (3.9), (3.10), and (3.11) give

$$(a \circ b) \circ c = a \circ (b \circ c). \tag{3.12}$$

Again by (iv) and (v) of Definition 3.1, given $0 \neq b \in D$, $T_D(b, \cdot)$ is one-to-one and onto D. Thus $\langle D, +, \circ \rangle$ is a skew field with identity 1. Also, if $(ab) \circ c = a \circ (b \circ c) = (a \circ b) \circ c$, then $ab = a \circ b$. Thus M' is a D-module if and only if "$\circ$" and "$\cdot$" are the same operation and $\langle D, +, \cdot \rangle$ is a skew field.

Conversely, take "$\circ$" and M' as above with $c = \sum_{i \in \Lambda} c_i \in M' = \dot{+}_{i \in \Lambda} D$. Then $((ab) \circ c)_i = (ab) \circ c_i = \bar{a} \circ (b \circ c_i) = (\bar{a} \circ b) \circ c_i$. Here we can solve for a given $\bar{a}$, b or for $\bar{a}$ given a, b in $ab = \bar{a} \circ b$ independent of c_i (as long as at least one $c_i \neq 0$). The solution will then give $(ab) \circ c = \bar{a} \circ (b \circ c)$ and M' is a uniform $\bar{D}$-module under "$\circ$".

Hence we have shown:

THEOREM 3.18. *If M is a uniform $\bar{D}$-module of dimension $|\Lambda| \geqslant 2$, then M is D-isomorphic to $\dot{+}_{i \in \Lambda} D$ where the module multiplication is given by* (3.4) *with "$\circ$" such that $\langle D, +, \circ \rangle$ is a skew field with identity 1. M is a D-module if and only if $\langle D, +, \cdot \rangle = \langle D, +, \circ \rangle$, in which case M is a vector space over $\langle D, +, \circ \rangle$. Conversely, if $\langle D, +, \circ \rangle$ is a skew field with identity 1, then* (3.4) *gives $\dot{+}_{i \in \Lambda} D$ as a uniform $\bar{D}$-module of dimension $|\Lambda|$.*

THEOREM 3.19. *Suppose* $\langle D, +, \circ\rangle$ *and* $\langle D, +, *\rangle$ *are skew fields making* $\dot{+}_{i\in\Lambda} D$ *a uniform* $\bar{D}$*-module as in Theorem 3.18. Then* $\dot{+}_{i\in\Lambda} D$ *under "*$\circ$*" is uniformly* $\bar{D}$*-isomorphic to* $\dot{+}_{i\in\Lambda} D$ *under "*$*$*" if and only if* $\langle D, +, \circ\rangle$ *is isomorphic to* $\langle D, +, *\rangle$.

Proof. Suppose $\sigma : \langle D, +, \circ\rangle \to \langle D, +, *\rangle$ is an isomorphism. Let $V = \dot{+}_{i\in\Lambda} D$ and define $\sigma : V \to V$ by $[\sigma(\sum_{i\in\Lambda} c_i)]_j = \sigma(c_j)$. Clearly, σ is additive, one-to-one, and onto V.

$$\left[\sigma\left(a \circ \left(\sum_{i\in\Lambda} c_i\right)\right)\right]_j = \left[\sigma\left(\sum_{i\in\Lambda} (a \circ c_i)\right)\right]_j = \sigma(a \circ c_j) = \sigma(a) * \sigma(c_j)$$

$$= \sigma(a) * \left[\sigma\left(\sum_{i\in\Lambda} c_i\right)\right]_j = \left[\sigma(a) * \sigma\left(\sum_{i\in\Lambda} c_i\right)\right]_j.$$

Thus σ is a uniform $\bar{D}$-isomorphism of $\dot{+}_{i\in\Lambda} D$ under "$\circ$" onto $\dot{+}_{i\in\Lambda} D$ under "$*$".

Now suppose $T : V \to V$ is a uniform $\bar{D}$-isomorphism of V under "$\circ$" onto V under "$*$". Pick a basis, $[\epsilon_i]_j = \delta_{ij}$ for V. Fix $i \in \Lambda$.

Suppose for $a \in D$, $\bar{a}$ is such that $T(a \circ \epsilon_i) = \bar{a} * T(\epsilon_i)$. Define $\sigma : \langle D, +, \circ\rangle \to \langle D, +, *\rangle$ by $\sigma(a) = \bar{a}$. Again it is clear that σ is additive, one-to-one, and onto D. By the definition of σ and Lemma 3.17 we have

$$T((a \circ b) \circ \epsilon_i) = \sigma(a \circ b) * T(\epsilon_i), \tag{3.13}$$

$$T(b \circ \epsilon_i) = \sigma(b) * T(\epsilon_i), \tag{3.14}$$

and

$$\begin{aligned} T((a \circ b) \circ \epsilon_i) &= T(a \circ (b \circ \epsilon_i)) = \sigma(a) * T(b \circ \epsilon_i) \\ &= \sigma(a) * (\sigma(b) * T(\epsilon_i)) \\ &= (\sigma(a) * \sigma(b)) * T(\epsilon_i). \end{aligned} \tag{3.15}$$

Then (3.13) and (3.15) give $\sigma(a \circ b) = \sigma(a) * \sigma(b)$ so that σ is an isomorphism of $\langle D, +, \circ\rangle$ onto $\langle D, +, *\rangle$.

COROLLARY 3.20. *If* $\langle F, +, \cdot\rangle$ *is a finite field and* M *is a uniform* $\bar{F}$*-module of dimension* $|\Lambda| \geqslant 2$*, then* M *is uniformly* $\bar{F}$*-isomorphic to the vector space* $\dot{+}_{i\in\Lambda} F$.

THEOREM 3.21. *If* D *is a unique left-division ring with identity 1, then* D *always has a uniform* $\bar{D}$*-module that is not a* D*-module except when* D *is* Z_p *or* Q.

Proof. Let P be the prime subfield of D generated by 1.

Case 1. Suppose that the dimension of D over P is infinite. If D is not associative, we may adjoin the proper number of indeterminants to P to obtain the dimension of D and form the field $P(x_i \mid i \in \Gamma) = \langle D, +, \circ \rangle$ where "$\circ$" and "$\cdot$" are distinct operations. Then $\dot{+}_{i \in \Lambda} D$ under "$\circ$" is a uniform $\bar{D}$-module that, by Theorem 3.18, is not a D-module. If D is associative, we can always find a different skew field having the same dimension over P as D which gives $\langle D, +, \circ \rangle$ where "$\circ$" is distinct from "$\cdot$".

Case 2. Suppose that the dimension of D over P is finite but not 1. If D is not associative, we can take a field $\langle D, +, \circ \rangle$ over P to give a uniform $\bar{D}$-module $\dot{+}_{i \in \Lambda} D$ under "$\circ$". If D is associative we can find a non-singular linear transformation T of D over P such that $T(1) = 1$ but T is not an automorphism of $\langle D, +, \cdot \rangle$. Define $\circ : D \times D \to D$ by $a \circ b = T^{-1}(T(a)\, T(b))$. $\langle D, +, \circ \rangle$ is a skew field with identity 1 and "$\circ$" is not the same operation as "$\cdot$".

Remark. Though Corollary 3.20 says that every uniform $\bar{F}$-module of dimension $|\Lambda| \geqslant 2$ over a finite field F is uniformly $\bar{F}$-isomorphic to the vector space $\dot{+}_{i \in \Lambda} F$, Theorem 3.21 says that there are uniform $\bar{F}$-modules that are not vector spaces when F is not Z_p.

3.3. *Classification of Irreducible Uniform $\bar{D}_n$-Modules, $2 \leqslant n < \infty$*

Throughout Section 3.3, D will denote a unique left-division ring $\langle D, +, \cdot \rangle$ with identity 1 and D_n the n by n matrices over D, where $e_i = (a_{jk})$ with $a_{jk} = \delta_{ij}\delta_{ik}$, $1 \leqslant i \leqslant n$.

THEOREM 3.22. *If M is an irreducible $\bar{D}_n$-module, $M \neq \{0\}$, then $\langle D, +, \cdot \rangle$ is a skew field and M is a D_n-module.*

Proof. Let $R = D_n$ and suppose that M is a non-zero irreducible $\bar{R}$-module. By Definition 3.2, M is a uniform $\bar{R}$-module.

There is an i with $1 \leqslant i \leqslant n$ such that $e_i M \neq \{0\}$. Without loss of generality we may assume that $e_1 M \neq \{0\}$. Then there is an $m \in M$ such that $e_1 m \neq 0$. Then $(Re_1)\, m$ is a non-zero submodule of M so that $(Re_1)\, m = M$.

By Lemma 2.32, parts (b) and (c) we have

$$(re_i)\, n = r(e_i n), \tag{3.16}$$

$$(e_i r)\, n = e_i(rn), \tag{3.17}$$

for all $r \in R$, $n \in M$.

By (3.16) and the remark above $M = (Re_1)\, m = ((Re_1)\, e_1)\, m = (Re_1)\,(e_1 m)$ and $(Re_j)\,(e_1 m) = ((Re_j)\, e_1)\, m = \{0\}$ for $j \neq 1$.

Let $R_{ij} = e_i R e_j$, $a \in R_{ij}$, $b \in R_{kq}$. Then, $(ab)\,(e_1 m) = 0$ unless $j = k$

and $q = 1$. Hence we will suppose that $a \in R_{ij}$, $b \in R_{j1}$. Now we have $(ab)\,(e_1m) = \bar{a}(b(e_1m))$ for some $\bar{a} \in Re_j$. Since $a \in R_{ij}$, by (3.17)

$$e_i((ab)\,(e_1m)) = (ab)\,(e_1m) = e_i(\bar{a}(b(e_1m))) = (e_i\bar{a})\,(b(e_1m))$$

and $e_\ell((ab)\,(e_1m)) = 0$ for $\ell \neq i$ so that $\bar{a} \in R_{ij}$. Conversely, for $\bar{a} \in R_{ij}$ we may choose $a \in R_{ij}$ for $(ab)\,(e_1m) = \bar{a}(b(e_1m))$.

Suppose $\bar{a}(b(e_1m)) = (ab)\,(e_1m) = (a'b)\,(e_1m)$, where $\bar{a}$, a, $a' \in R_{ij}$, $b \in R_{j1}$ are non-zero. Then $ab - a'b \in Re_1 \cap \ell\text{-ann}\,(e_1m)$. But Re_1 is a minimal left ideal of R and, by Lemma 3.11, ℓ-ann (e_1m) is a left ideal of R with $e_1 \notin \ell\text{-ann}\,(e_1m)$. Hence $ab - a'b = 0$. Then $(a - a')\,b = 0$. But the only element of Re_j that annihilates a non-zero element of e_jRe_1 is 0 and $a = a'$. Thus, given b non-zero in R_{j1}, the solution for $a \in R_{ij}$ in $\bar{a}(b(e_1m)) = (ab)\,(e_1m)$ is unique.

Now we have $T_{R_{ij}} : R_{j1} \times R_{ij} \to R_{ij}$ and $\circ_{ij} : R_{ij} \times R_{j1} \to R_{i1} \subseteq R$ as given in Lemma 3.12. We will denote $T_{R_{ij}}$ by T_{ij}. We may consider T_{ij} as mapping $D \times D \to D$ and "$\circ_{ij}$" as an operation on $D \times D$ to D, since each $a \in R_{ij}$ is actually an element of D in the (ij)th position of an n by n matrix. We may also consider elements of Re_1 as column vectors.

Now define $\circ : R \times Re_1 \to Re_1$ by $(a_{ij}) \circ (b_{k1}) = (\sum_{k=1}^{n} a_{ik} \circ_{ik} b_{k1})$. By the distributivity of "$\circ_{ik}$" given in Lemma 3.12 "$\circ$" is a distributive operation on $R \times Re_1$.

By (3.16) and (3.17) we have $T_{j1}(e_1, a) = a$ and $T_{11}(r, e_1) = e_1$ for $r \in R_{i1}$, $a \in R_{j1}$.

Define $\sigma : M \to Re_1$ by $\sigma(r(e_1m)) = re_1$. By the uniqueness of solvability given above, $re_1 = se_1$ implies that $r(e_1m) = s(e_1m)$ so that σ is well-defined. If $re_1 = 0$, $r(e_1m) = 0$, so that σ is one-to-one. σ is onto Re_1 since $re_1 \in Re_1$ gives $(re_1)\,(e_1m) \to (re_1)\,e_1 = re_1$. Clearly σ is additive.

Now we wish to show that σ is an R-isomorphism of M onto Re_1 under "$\circ$":

$$\begin{aligned}
\sigma(a(r(e_1m))) &= \sigma\left(\left(\sum_{1\leqslant i,j\leqslant n} e_iae_j\right)\left(\left(\sum_{k=1}^{n} e_kre_1\right)(e_1m)\right)\right)\\
&= \sigma\left(\left(\left(\sum_{1\leqslant i,j\leqslant n} \overline{e_iae_j}\right)(e_jre_1)\right)(e_1m)\right)\\
&= \sigma\left(\left(\sum_{1\leqslant i,j\leqslant n} T_{ij}(e_jre_1\,,\, e_iae_j)\,(e_jre_1)\right)(e_1m)\right)\\
&= \left(\sum_{j=1}^{n} T_{ij}(e_jre_1\,,\, e_iae_j)\,(e_jre_1)\,e_1\right)\\
&= \left(\sum_{j=1}^{n} T_{ij}(e_jre_1\,,\, e_iae_j)\,(e_jre_1)\right) = a \circ \sigma(r(e_1m)).
\end{aligned}$$

Now we wish to show that "$\circ_{ij}$" must be the same operation as "$\cdot$" for all i, j.

Fix k with $1 \leqslant k \leqslant n$ and let $A = (a_{ik})$, $\bar{A} = (\bar{a}_{ik})$, $B = (b_{kj})$, $C = (x_{j1})$. We obtain

$$(AB) \circ C = \begin{pmatrix} (a_{1k}b_{k1}) \circ_{11} x_{11} + (a_{1k}b_{k2}) \circ_{12} x_{21} + \cdots + (a_{1k}b_{kn}) \circ_{1n} x_{n1} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ (a_{nk}b_{k1}) \circ_{n1} x_{11} + (a_{nk}b_{k2}) \circ_{n2} x_{21} + \cdots + (a_{nk}b_{kn}) \circ_{nn} x_{n1} \end{pmatrix} \tag{3.18}$$

$$\bar{A} \circ (B \circ C) = \begin{pmatrix} \bar{a}_{1k} \circ_{1k} (b_{k1} \circ_{k1} x_{11}) + \cdots + \bar{a}_{1k} \circ_{1k} (b_{kn} \circ_{kn} x_{n1}) \\ \vdots \qquad\qquad \vdots \\ \bar{a}_{nk} \circ_{nk} (b_{k1} \circ_{k1} x_{11}) + \cdots + \bar{a}_{nk} \circ_{nk} (b_{kn} \circ_{kn} x_{n1}) \end{pmatrix}. \tag{3.19}$$

Necessary conditions for the solution of

$$(AB) \circ C = \bar{A} \circ (B \circ C) \tag{3.20}$$

for $\bar{A}$ given B, C non-zero are

$$\sum_{i=1}^{n} b_{ki} \circ_{ki} x_{i1} = 0 \qquad \text{implies for all} \quad q \quad \text{and}$$

$$\text{all} \quad a_{qk} \quad \text{that} \quad \sum_{i=1}^{n} (a_{qk}b_{ki}) \circ_{qi} x_{i1} = 0. \tag{3.21}$$

Consider (3.21) when $k = 1$ and all $x_{i1} = 0$, except for $i = 1, t$ where $t \neq 1$. (3.21) becomes

$$b_{11} \circ_{11} x_{11} + b_{1t} \circ_{1t} x_{t1} = 0 \qquad \text{implies for all} \quad q$$

$$\text{and all} \quad a_{q1} \quad \text{that} \quad (a_{q1}b_{11}) \circ_{q1} x_{11} + (a_{q1}b_{1t}) \circ_{qt} x_{t1} = 0. \tag{3.22}$$

Now the procedure is the same as found in Theorem 2.35 showing

$$\circ_{qt} = \circ_{1t}, \qquad t \neq 1$$

$$\circ_{qt} = \cdot, \qquad t \neq 1$$

$$\circ_{q1} = \circ_{qt}, \qquad t \neq 1.$$

Hence, we have that all "$\circ_{ij}$" are the same as "$\cdot$". Now we will show that "$\cdot$" must be associative.

Equation (3.22) now gives $ax + by = 0$ implies, for all $c \in D$, $(ca)\,x + (cb)\,y = 0$. Take $y = 1$ and $ax = b \cdot 1$. Then, for all $c \in D$, $(ca)\,x = (cb) \cdot 1 = c(b \cdot 1) = c(ax)$ and D is associative.

Hence the theorem has been established.

3.4. *Classification of $\bar{R}$-Modules where R is a Semi-simple Ring with Identity under Axiom 3 and* D.C.C.

First, we will consider the case where M is an $\bar{R}$-module, $R = D_n$, $2 \leqslant n < \infty$, and D a unique left-division ring with identity 1. Write $M = \sum_{i \in \Lambda} M_i$ where the M_i are uniform $\bar{R}$-modules.

LEMMA 3.23. *For each $i \in \Lambda$ and $m_i \in M_i$, $(Re_j)\,m_i$ is an irreducible uniform $\bar{R}$-module.*

Proof. Identical to the associative case.

THEOREM 3.24. *If M is a non-zero $\bar{R}$-module, $R = D_n$, $2 \leqslant n < \infty$, and D a unique left-division ring with identity, then D is a skew field and M is an R-module.*

Proof. Write $M = \sum_{i \in \Lambda} M_i$, a sum of uniform $\bar{R}$-modules. Then

$$M = \sum_{i \in \Lambda} M_i = \sum_{i \in \Lambda} \sum_{m \in M_i} Rm = \sum_{i \in \Lambda} \sum_{m \in M_i} \left(\sum_{1 \leqslant j \leqslant n} (Re_j)\,m \right)$$

where $(Re_j)\,m$ is an irreducible $\bar{R}$-module for some $i \in \Lambda$, some $m \in M_i$, some j with $1 \leqslant j \leqslant n$, $(Re_j)\,m \neq \{0\}$. By Theorem 3.22, D is a skew field and each $(Re_j)\,m$ is a D_n-module so that M is a D_n-module.

Now we consider the case where M is a $\bar{D}$-module, D a unique left-division ring with identity 1. Write $M = \sum_{i \in \Lambda} M_i$ a sum of uniform $\bar{D}$-modules. Using the multiplications given by the components above we have the following:

THEOREM 3.25. *We may write $M = \sum_{i \in \Lambda_0} \oplus M_i \oplus \sum_{i \in \Lambda_1} \oplus M_i$ where $i \in \Lambda_0$ implies that M_i is a one-dimensional uniform $\bar{D}$-module with multiplication "$\circ_i$" such that $\langle D, +, \circ_i \rangle$ is a unique left-division with identity 1 which is not associative and $i \in \Lambda_1$ implies M_i is a uniform $\bar{D}$-module with multiplication "$\circ_i$" such that $\langle D, +, \circ_i \rangle$ is a skew field. If we also write*

$$M = \sum_{i \in \Gamma_0} \oplus N_i \oplus \sum_{i \in \Gamma_1} \oplus N_i$$

as above, then there is a one-to-one onto mapping $T : \Lambda_0 \cup \Lambda_1 \to \Gamma_0 \cup \Gamma_1$ such that $T(\Lambda_0) = \Gamma_0$ and $T(\Lambda_1) = \Gamma_1$ with $M_i = N_{T(i)}$ and "$\circ_i$" is the same operation as "$\circ_{T(i)}$".

Now we consider the case where R is a semi-simple ring with identity under Axiom 3 and D.C.C.

LEMMA 3.26. *Suppose $R = R_1 \oplus R_2$, a ring direct sum, where R has identity 1. Suppose M is an $\bar{R}$-module. Then $M = R_1M \oplus R_2M$ where R_iM is an $\bar{R}(\bar{R}_i)$-module.*

Proof. Write $M = \sum_{j \in \Lambda} M_j$, a sum of uniform $\bar{R}$-modules. Let $1 = e_1 + e_2$, $e_i \in R_i$.

Let $x_i \in R_iM = \sum_{j \in \Lambda} R_iM_j$. Then, for $a_q \in R_q$, $a_q x_i = \sum_{j \in \Lambda} a_q(r_{ij}m_j) = 0$ for $q \neq i$. Hence $x_i = (e_1 + e_2)\, x_i = e_i x_i$ for $x \in R_iM$. $0 = x_1 + x_2$ implies that $0 = e_i x_i = x_i$ and the sum is direct.

THEOREM 3.27. *Suppose R is a semi-simple ring with identity satisfying Axiom 3 and* D.C.C. *Write $R = \sum_{i=1}^{n} \oplus R_i$, where the R_i are the simple components of R. If M is an irreducible $\bar{R}$-module, then for some $1 \leqslant i \leqslant n$, $R_iM = M$ and $R_jM = \{0\}$ for $j \neq i$, so that M is an irreducible $\bar{R}_i$-module.*

Proof. This follows by Lemma 3.26.

THEOREM 3.28. *Suppose R is a semi-simple ring with identity under Axiom 3 and* D.C.C. *Suppose M is an $\bar{R}$-module. Then $M = \sum_{i=1}^{n} \oplus R_iM$, where R_iM is classified by Theorem 3.25 for R_i a unique left-division ring or R_iM is an R_i-module for $R_i = D_n$, D (depending upon i) some skew field.*

We finish with a theorem concerning radicals under Axiom 3 in connection to $\bar{R}$-modules.

THEOREM 3.29. *Suppose R has an identity and satisfies Axiom 3 and* D.C.C. *Let $N = \operatorname{rad} R$ and suppose that M is an $\bar{R}$-module. Then M is completely reducible if and only if $NM = \{0\}$.*

Proof. Suppose that M is completely reducible and $M = \sum_{i \in \Lambda} \oplus M_i$, where the M_i are irreducible (hence uniform) $\bar{R}$-submodules of M. To show that $NM = \{0\}$ it suffices to show that $NM_i = \{0\}$ for each $i \in \Lambda$. Hence we may assume that M is an irreducible (uniform) $\bar{R}$-module and show that $NM = \{0\}$.

NM is an $\bar{R}$-module so that $NM = M$ or $NM = \{0\}$. Suppose $NM = M$. Take an integer i such that $N^iM \neq \{0\}$, $N^{i+1}M = \{0\}$. Such an integer exists since N is nilpotent. Then, for some $m \in M$, $N^im \neq \{0\}$ so that $N^im = M$. Hence, for some $n \in N^i$, $nm = m$ and $(n - 1)\, m = 0$. Then

$$(n - 1)(n - 1) = n^2 - 2n + 1 = n^2 - n + (1 - n).$$

But

$$((n-1)(n-1))\,m = ((n^2-n)+(1-n))\,m = -\,nm = -\,m$$

and

$$((n-1)(n-1))\,m = \overline{(n-1)}\,((n-1)\,m) = 0.$$

This is a contradiction. Hence, $NM = \{0\}$.

Conversely, suppose $NM = \{0\}$. Then M is an $\overline{R/N}$-module and the theorem follows from Theorems 3.28, 3.25, and the associative theory.

To conclude Section 3, we remark that many module theoretic results, e.g. those in [*1*], may be extended to $\bar{R}$-modules, R an arbitrary non-associative ring with identity. However, the preceding results in this section imply that for many such R, no $\bar{R}$-modules, other than $\{0\}$, exist.

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